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Grassmann–Cartan Algebra and Klein–Gordon Equations

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Given a Riemannian structure (M, g), a hypothesis is investigated that if $\alpha = \oplus_{p=0}^{n} \alpha_p \in \Lambda(M)$ is submitted to the differential condition $(g + \delta + \kappa)\alpha = 0$, $\kappa = mc/\hbar$—which implies that each component of α fulfills the Klein–Gordon equation $(\Delta - \kappa^2)\alpha_p = 0$, α ought to be interpreted as a natural complex of the bosonic fields. Then it is found that the complex α admits the interpretation in the sense of first quantization with $\Lambda(M)$ being a convex set of states, with the structure of a Hilbert space over $\mathscr{R}$. The definite spin states of bosons are then pure states which are not conserved by the temporal evolution.

1. INTRODUCTION

The ideas of this paper and its results should perhaps be identified as belonging to the realm of a frontier between differential geometry and the foundations of relativistic quantum mechanics.

It is well known that the Klein–Gordon equation (K. G. subsequently), fails to serve as an adequate relativistic wave equation. Nevertheless, according to the Yukawa idea, it does serve as the starting point of the quantum field theory (second quantization) of various sorts of the bosonic massive particles (usually treated in the case of the vector bosons according to the Proca scheme, with the auxiliary Lorentz condition). However, according to Dirac's deep remark, the so understood bosons which *do not* possess a relativistic wave equation (in the sense of first quantization) are really not the "true" particles in contrast to the fermions: for these the ingenious Dirac's idea of executing the root $(p_x^2 + p_y^2 + p_z^2 + m^2c^2)^{1/2}$ by employing

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4×4 matrices has provided for the spin $s = 1/2$ the prototypic correct relativistic wave equation. The theoretical physicists aware of this, long ago proposed some alternative treatments of bosons: here belongs the quantum mechanical interpretation of the photon with the wave function $f_{\mu\nu} = f_{[\mu\nu]} \in \mathscr{C}$ (Akhiezer and Berestetskii, 1965), the Duffin–Kemmer scheme (Duffin, 1938; Kemmer, 1939), etc. The excellent book by the late E. M. Corson on the relativistic wave equations (Corson, 1953) can be perhaps considered a "summa theologica" of the early attempts in this direction, covering the numerous pertinent references. More recently, K. G. equations for the complex scalar field have been studied from the point of view of the spectral theory, employing as a scalar product in an associated Hilbert space, the "energy scalar product" whose existence relies on the positive-definite energy integral of the complex scalar boson; see Weder (1977, 1978) for more actual references to the subject. Notice also that Weder (1977) indeed suggests an interesting interpretation in the sense of the first quantization for a complex scalar boson (Weder, 1977, p. 115), although the problem of the meaning of its covariance with respect to the Poincaré group seems to remain open. However, none of these attempts has achieved the status of a generally accepted physical theory which corresponds to the experimental facts, apart from the appreciation of the interesting nature of the mathematical structures considered.

Now, the main thrust of the present paper consists in calling attention to the fact that the canonical concepts of the differential geometry, the external differential d, the Hodge star $*$, and the induced by these codifferential δ, offer us in a sense a general alternative to the standard "Diracization" process of the Klein–Gordon equations.

The basic idea of this alternative is this: let $V = (M, g)$ be a Riemannian space over a finite-dimensional differential manifold M, $\mathrm{Dim}(M) = n < \infty$, with a nonsingular metric g, carrying automatically at each point $p \in M$ the graded Grassmann–Cartan algebra $\Lambda = \oplus_{p=0}^{n} \Lambda^p$ with its operations $(\cdot \mathfrak{F}, +, \wedge)$ (the field $\mathfrak{F}$ being usually constrained to $\mathscr{R}$ or $\mathscr{C}$), and, as we deal with a metrical structure, additionally endowed with the Hodge isomorphism $*: \Lambda^p \Leftrightarrow \Lambda^{p'}$, $p + p' = n$.

Then, d and δ are the differential mappings:

$$\begin{aligned} &d: \Lambda^p \to \Lambda^{p+1}, \qquad d(\Lambda^n) = 0, \qquad d^2 = 0 \\ &\delta: \Lambda^p \to \Lambda^{p-1}, \qquad \delta(\Lambda^0) = 0, \qquad \delta^2 = 0 \end{aligned} \tag{1}$$

the nil-potence of d and δ being equivalent to the Poincaré lemma. Now, the Laplace–Beltrami operator (i.e., the generalized Laplace or d'Alambert operator) $\Delta : \Lambda^p \to \Lambda^p$ defined as

$$\Delta : d\delta + \delta d \tag{2}$$

and understood as a differential mapping *on the whole* $\Lambda = \oplus_{p=0}^{n} \Lambda^p$, can be interpreted as:

$$\begin{aligned}\Delta &= d\delta + \delta d \\ &= (d+\delta)^2 \qquad (\text{because } d^2 = 0 = \delta^2)\end{aligned} \tag{3}$$

Hence, "$d + \delta$" is a "natural root" of Δ understood as the mapping $\Lambda \to \Lambda$.

Consequently, if we make the decisive step of subjecting *all* components of $\alpha = \oplus_{p=0}^{n} \alpha_p \in \Lambda$ to the "Klein–Gordon equations":

$$(\Delta - \kappa^2)\alpha_p = 0, \qquad \kappa = \text{const} \tag{4}$$

these equations are equivalent to

$$(d + \delta - \kappa)(d + \delta + \kappa)\alpha = 0 \tag{5}$$

and, if we postulate then the *stronger* differential condition (of *the first differential* order):

$$(d + \delta + \kappa)\alpha = 0 \tag{6}$$

as its consequence, the K. G. equations (4) are automatically fulfilled.

The procedure outlined above is exactly analogous to the standard Diracization process: indeed, if all components of the wave function ψ_k have to be compatible with the special-relativistic energy-momentum conservation (i.e., fulfill K. G. equations):

$$\left[p^\mu p_\mu - (mc)^2\right]\psi_k = 0 \tag{7}$$

meaning by p_μ the standard quantum mechanical operators [signature assumed $(+, -, -, -)$], then these equations are equivalent to

$$(\gamma^\mu p_\mu - mc)(\gamma^\nu p_\nu + mc)\psi = 0 \tag{8}$$

where ψ is a column of ψ_k's on which the rectangular matrices γ^μ can act, these being submitted to the conditions

$$\gamma_\mu p_\nu - p_\nu \gamma_\mu = 0, \qquad \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu} \tag{9}$$

Then the *stronger* condition:

$$\left(\gamma^{\mu} p_{\mu} + mc\right)\psi = 0 \tag{10}$$

which assures the validity of K. G. equations (7), with γ^{μ}'s realized as 4×4 matrices (i.e., the *minimal* dimension of these compatible with the anticommutation rule), amounts just to the Dirac equations in their standard form.

The (4) equations will be seen in the next section in the case of the flat four-dimensional metric of the hyperbolic signature $(+,-,-,-)$ just equivalent to the standard K. G. equations if we identify

$$\kappa = mc/\hbar \tag{11}$$

m being the inertial mass of the considered field theoretical complex. Thus the crucial equations of this paper, (6), can be understood also as a *stronger* form—"*á la* Dirac"—of the demand that each component of the field theoretical complex is compatible with the special relativistic energy-momentum conservation $p^{\mu}p_{\mu} - (mc)^2 = 0$.

It should be pointed out, however, that the mathematical idea of taking seriously "$d + \delta$" as a natural root of Δ understood as an operator on whole Λ, is a very general one: it applies for (M, g) of arbitrary dimensionality, the case of the (complexified) analytic structures, "complex relativity" (Plebański and Schild, 1976; Boyer et al., 1980), being implicitly included. When the structure (M, g) is real, and the signature has an objective geometrical meaning, the "Diracization" of (4) to the first-order equations (6) still applies for an arbitrary signature. This opens a way—departing from the hyperbolic metric—also to the natural study of the "instantonic" aspect of the structure $(d + \delta + \kappa)\alpha = 0$.

The first formal idea of this paper "$\sqrt{\Delta} = \Delta + \delta''$: $\Lambda \to \Lambda$, is complemented by this second basic ingredient: a suggestion that $\Lambda(M)$ which admits the structure of a Hilbert space, can be perhaps considered as generating a quantum mechanical convex set of states (Mielnik, 1974).

Given a real structure (M, g) it is well known that if (i) the metric is positive definite and (ii) M is compact, then $\Lambda(M)$ is indeed a Hilbert space over $\mathscr{R}$, with the scalar product between $\alpha = \oplus_{p=0}^{n}\alpha_p$, $\beta = \oplus_{p=0}^{n}\beta_p \in \Lambda(M)$ defined by

$$(\alpha, \beta) := \int_M * \sum_{p=0}^{n} \alpha_p \lrcorner \beta_p \tag{12}$$

in the sense of which d and δ are the mutually co-adjoint operators, $(d\alpha, \beta) = (\alpha, \delta\beta)$. Notice that the assumption of compactness for M is not very essential—it can be replaced by constraining the admissible "states" α to be square integrables", $(\alpha, \alpha) < \infty$, with the components of α_p's "vanishing at infinity" if, e.g., $M = \mathscr{R}^n$.

However, except perhaps for the interpretation in the spirit of instantons theory, such "states" α without the temporal evolution seem to be of little use. On the other hand, if the real $V_n = (M_n, g_n)$ has the structure of $V_1 \times V_{n-1}$, $g_1 = dt \otimes_s dt$ trivial, $M_n = \mathscr{R} \times M_{n-1}, g_{n-1}$ positive definite, the n-dimensional metric

$$g_n = g_1 - g_{n-1} \tag{13}$$

being thus hyperbolic, we will see that there is a *natural* manner of interpreting $\alpha \in \Lambda(M_n)$ as the set of "states" *at different time slices*, $\alpha(t) \in \Lambda(M_{n-1})$, their evolution being governed by our invariant equation $(d + \delta + \kappa)\alpha = 0$, while they do form a Hilbert space with $\langle \alpha'(t'), \alpha''(t'') \rangle$ constructed as in (12) *but in the sense of* (M_{n-1}, g_{n-1}).

The basic ideas of this paper have been outlined in this section in the concise language of the contemporary differential geometry. For the benefit of the readers rather accustomed to work with physical fields as described in terms of the local charts, the following section will provide a concise resumé of the definitions of the concepts like $*$, $\lrcorner$, d, δ, and Δ in terms of local coordinates.[2]

2. LOCAL DEFINITIONS AND THE GEOMETRIC CONCEPTS USED

Let in a coordinates chart $\{x^\mu\}$, $\mu = 1, \ldots, n$, the metric of an n-dimensional Riemannian structure be given in the conventional form

$$g = g_{\mu\nu}\, dx^\mu \underset{s}{\otimes} dx^\nu = \eta_{ab} e^a \underset{s}{\otimes} e^b \tag{14}$$

where $\otimes_s$ denotes the symmetric tensor product, $g_{\mu\nu} = g_{(\mu\nu)}$ is nonsingular, $\eta_{ab} = \eta_{(ab)}$ is the signature metric (constant and numeric; $a, b, \ldots = 1, \ldots n$), while $e^a = e^a_\mu\, dx^\mu$ constitute the "n-Bein" of 1-forms. Without any generality lost, with M and g being either real or complex analytic (in the last case $g_{\mu\nu}$ is holomorphic in x^μ), the signature metric can be assumed in the

[2] For these readers who are accustomed with these concepts as outlined, say, in the Flanders book (Flanders, 1963), it is recommended to skip the next section, and to follow the text from Section 3, while checking back the definitions of Section 2, only when these are involved in the local arguments of the essential text.

form of

$$\|\eta_{ab}\| = \|\mathrm{diag}\Big(\underbrace{1,\ldots,1}_{n = n_{(+)}}, \; + \; \underbrace{-1,\ldots,-1}_{n_{(-)}}\Big)\| \tag{15}$$

Of course, in the complex analytic case $n_{(-)}$ can be chosen as convenient, $n_{(+)} = n - n_{(-)}$, while in the case of the real structure the signature has an objective geometrical meaning.

Now, a general element $\alpha \in \Lambda \equiv$ Grassmann–Cartan algebra, has the form of $\alpha = \oplus_{p=0}^{n} \alpha_p$ ($\oplus$ denotes the direct sum) with α_p's given in local coordinates as

$$\alpha_p = \frac{1}{p!} \alpha_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \in \Lambda^p \tag{16}$$

In other words, α amounts to a set of *skew* tensors in the standard sense, which can be understood as organized in the form of a column

$$\begin{pmatrix} \alpha_0 \\ \alpha_{\mu_1} \\ \cdots \\ \alpha_{\mu_1 \cdots \mu_p} \\ \cdots \\ \alpha_{\mu_1 \cdots \mu_n} \end{pmatrix} \tag{17}$$

These tensors with $2^n = \Sigma_{p=0}^{n} \binom{n}{p}$ of independent components are functions of x^μ, while $\{x^\mu\}$ covers an open domain $\mathscr{D}(M)$ of the manifold M.

Then, in terms of these concepts, the Hodge star $*$ or the generalized duality operation, is an isomorphism $* : \Lambda^p \Leftrightarrow \Lambda^{p'}$, $p + p' := n$, defined by

$$\begin{aligned} \Lambda^p \ni \alpha_p &\to \Lambda^{p'} \ni * \alpha_p \\ &:= \frac{1}{p!\,p'!} \sqrt{/}\, \varepsilon^{\mu_1 \cdots \mu_p}{}_{\cdot \; \cdot \;}{}_{\nu_1 \ldots \nu_{p'}} \alpha_{\mu_1 \ldots \mu_p} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{p'}} \\ \times \sqrt{/} &:= \left[\det(g_{\mu\nu})/\det(\eta_{ab})\right]^{1/2} \end{aligned} \tag{18}$$

This canonical definition applies also with $\|\eta_{ab}\|$ chosen as convenient, i.e., not necessarily constrained to the diagonal form (15). Of course, $\varepsilon_{\mu_1 \ldots \mu_n} = \varepsilon_{[\mu_1 \ldots \mu_n]}$, $\varepsilon_{12 \ldots n} = 1$, denotes here the Levi-Cività symbol; the "fat dots" denote the original places of indices which have been raised by means of the contravariant metric. If $\alpha_{\mu_1 \ldots \mu_p}$ is a tensor then in $* a_p$ the coefficient at $dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{p'}}$ is a pseudo-tensor; a coordinate transformation $x^\mu = x^\mu(x')$ which changes the relative orientation of the chart with respect to the "n-Bein" e^a, affects it—apart from the usual tensorial transformation—by the factor sign $[\partial(x)/\partial(x')]$. The so-defined star operation has the crucial property:

$$\Lambda^p \ni \alpha_p \to * * \alpha_p = \det(\eta^{ab})(-1)^{pp'} \alpha_p \tag{19}$$

Next, the contraction operation $\lrcorner$ (step), or the inner product, it is convenient to define for our purposes as the mapping $\lrcorner : \Lambda^q \times \Lambda^p \to \Lambda^{p-q}$ described in terms of local components as

$$\alpha_q \lrcorner \beta_p := \frac{1}{q!(p-q)!} \alpha^{\mu_1 \cdots \mu_q} \beta_{\mu_1 \ldots \mu_q \nu_1 \ldots \nu_{p-q}} dx^{\nu_1} \ldots dx^{\nu_{p-q}} \tag{20}$$

Notice that

$$\alpha_q \lrcorner \beta_p = \det(\eta^{ab})(-1)^{p'(p-q)} *(\alpha_q \wedge * \beta_p) \tag{21}$$

Now, as far as the differential operations are concerned, the external differential $d : \Lambda^p \to \Lambda^{p+1}$ is defined in terms of the local components of α_p by

$$\Lambda^p \ni \alpha_p \to d\alpha_p := \frac{1}{p!} \alpha_{\mu_1 \ldots \mu_p, \lambda} dx^\lambda \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$$

$$= \frac{(-1)^p}{p!} \alpha_{[\mu_1 \cdots \mu_p, \mu_{p+1}]} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p+1}} \tag{22}$$

The nil-potence of d, $d^2 = 0$, is obvious from this definition. For $p = 0$, $d\alpha_0 = \alpha_{0,\mu} dx^\mu$ amounts just to the standard gradient and for $p = 1$, $d(\alpha_\mu dx^\mu) = -\alpha_{[\mu,\nu]} dx^\mu \wedge dx^\nu$ has the meaning of the rotation, being thus for arbitrary p a generalized rotation.

Then, the notion of codifferential $\delta : \Lambda^p \to \Lambda^{p-1}$, $(\delta \Lambda^0 = 0)$, is defined by

$$\Lambda^p \ni \alpha_p \to \delta\alpha_p := \det(\eta_{ab})(-1)^{np+n+1} * d * \alpha_p$$

$$= \frac{(-1)^p}{(p-1)!} \alpha_{\mu_1 \cdots \mu_{p-1} \rho ; \rho} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p-1}} \in \Lambda^{p-1} \tag{23}$$

The nil-potence of δ follows from this definition as a consequence of (19) and $d^2 = 0$. In the local expression for $\delta, (\ldots)^{;\rho} := (\ldots)_{;\sigma} g^{\sigma\rho}$, and "$;\sigma$" denotes just the usual covariant derivative defined in terms of the Christophel symbols $\{{}^{\alpha}_{\beta\gamma}\}$. The concept of δ thus amounts simply to the usual covariant divergence of a skew tensor.

Now, in order to understand the specific choice for the factor in the first line of (24), consider for $\alpha = \oplus^{n}_{p=0}\alpha_p$, $\beta = \oplus^{n}_{p=0}\beta_p$ being two arbitrary elements of Λ, the bilinear form:

$$\begin{aligned}\Lambda^n \ni \mathscr{I}[\alpha,\beta] &:= \sum_{p=0}^{n} \alpha_p \wedge *\beta_p = \mathscr{I}[\beta,\alpha] \\ &= *\left(\sum_{p=0}^{n} \alpha_p \lrcorner \beta_p\right) \\ &= \left(\sum_{p=0}^{n} \frac{1}{p!}\alpha^{\mu_1\cdots\mu_p}\beta_{\mu_1\cdots\mu_p}\right)\sqrt{/}\, dx^1 \wedge \cdots \wedge dx^n \end{aligned} \tag{24}$$

The invariant definition of δ is just so arranged that for *every* α, $\beta \in \Lambda$:

$$\begin{aligned}\mathscr{I}[d\alpha,\beta] - \mathscr{I}[\alpha,\delta\beta] &= d\left\{\sum_{p=0}^{n} \alpha_p \wedge *\beta_{p+1}\right\} \\ &= \text{an exact } n\text{-form}\end{aligned} \tag{25}$$

(Verifying this, one uses the Leibnitz property of the external differentiation, $\alpha \in \Lambda^p \to d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$.)

Notice that with $\mathscr{D} \in M$ being a domain of M and $\partial\mathscr{D}$ being its boundary, if we define

$$(\alpha,\beta)_{\mathscr{D}} := \int_{\mathscr{D}} \mathscr{I}[\alpha,\beta] \tag{26}$$

then according to Gauss–Stokes–Ostrogracki theorem

$$(d\alpha,\beta)_{\mathscr{D}} - (\alpha,\delta\beta)_{\mathscr{D}} = \int_{\partial\mathscr{D}} \sum_{p=0}^{n} \alpha_p \wedge *\beta_{p+1} \tag{27}$$

Finally, we shall provide the local representation of the Laplace–Beltrami operator $\Delta := d\delta + \delta d$. Let $R^{\alpha}_{\beta\gamma\delta}$ be the curvature tensor induced by $g_{\mu\nu}$, $R_{\alpha\beta} := R^{\rho}_{\alpha\beta\rho}$ the Ricci tensor, while $R := R^{\rho}_{\rho}$ is the scalar curvature. Then

$$R^{\alpha\beta}{}_{\gamma\delta} =: C^{\alpha\beta}{}_{\gamma\delta} + \frac{1}{n-2}\delta^{\alpha\beta\rho}_{\gamma\delta\sigma}C^{\sigma}_{\rho} + \frac{R}{n(n-1)}\delta^{\alpha\beta}_{\gamma\delta} \tag{28}$$

where $C_{\alpha\beta} := R_{\alpha\beta} - (1/n)g_{\alpha\beta}R$ is the traceless part of the Ricci tensor, and $\delta^{\cdots}{}_{\cdots}$ are the generalized Kronecker δ's defines the totally traceless conformal curvature tensor $C^{\alpha}_{\beta\gamma\delta}$. In terms of these concepts one easily works out that

$$\begin{aligned}\Lambda^{p} \ni \alpha_{p} &\to \Delta\alpha_{p}\\ &= -\frac{1}{p!}\left\{\alpha_{\mu_1\ldots\mu_p;\rho}{}^{;\rho} + p\sum_{i=1}^{p-1}\alpha_{\mu_1\ldots\lambda_i\ldots\mu_{p-1}\rho}C^{\lambda_i}{}_{\mu_i}{}^{\rho}{}_{\mu_p}\right.\\ &\qquad\left. + \frac{p(n-2p)}{n-2}\alpha_{\mu_1\ldots\mu_{p-1}\rho}C^{\rho}_{\mu_p} + \frac{p(n-p)}{n(n-1)}R\alpha_{\mu_1\ldots\mu_p}\right\}\\ &\quad\times dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\end{aligned} \tag{29}$$

Using this formula one should remember that for $n = 2,3$ $C^{\alpha}_{\beta\gamma\delta} \equiv 0$, and for $n = 2$ also $C_{\alpha\beta} \equiv 0$.

Consider now the $(\Delta - \kappa^2)\alpha_p = 0$ equations, $\kappa = mc/\hbar$, in the case of a flat space-time described in terms of the Cartesian coordinates:

$$g = dt \underset{s}{\otimes} dt - dx \underset{s}{\otimes} dx - dy \underset{s}{\otimes} dy - dz \underset{s}{\otimes} dz \tag{30}$$

By using (29) with $\{x^{\mu}\} = \{t, x, y, z\}$ we easily infer that they amount to

$$\left[-\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2 - \left(\frac{mc}{\hbar}\right)^2\right]\alpha_{\mu_1\ldots\mu_p} = 0, \qquad p = 0,\ldots,4 \tag{31}$$

i.e., precisely to the Klein–Gordon equations for *all* 16 of independent components of $\alpha_{\mu_1\ldots\mu_p}$. Thus, $(\Delta - \kappa^2)\alpha_p = 0$ equations with the hyperbolic signature, $n^+ = 1$, $n^- = n - 1$, indeed generalize the K. G. equations for the case of a curved space.

Although we described in this section $*$, $\lrcorner$, d, δ, and Δ as operations defined in terms of chart components of $\alpha_p \in \Lambda^p$, all these operations possess an intrinsic geometrical meaning over the *whole* M, independent of the choice for the local charts.

3. GENERAL PROPERTIES OF THE $(d + \delta + \kappa)\alpha = 0$ EQUATIONS

The ideas of Section 1 have led us to the hypothesis that given a Riemannian structure (M, g) it is of interest to consider an element of the Grassmann–Cartan algebra $\alpha = \oplus_{p=0}^{n}\alpha_p \in \Lambda(M)$, each of its components fulfilling the Klein–Gordon equation $(\Delta - \kappa^2)\alpha_p = 0$. When submitted to a *stronger* differential condition $(d + \delta + \kappa)\alpha = 0$, $\Rightarrow (\Delta - \kappa^2)\alpha_p = 0$, it is a natural dynamical field-theoretical complex of the bosonic fields. This section begins a systematical study of this hypothesis. Its objective is to outline the basic structural facts which follow when the "dynamical equation" $(d + \delta + \kappa)\alpha = 0$ is postulated, independently from any specific assumptions about the nature of (M, g).

The n-dimensional $(d + \delta + \kappa)\alpha = 0$ equations stated more explicitly amount to

$$\Lambda^p \ni d\alpha_{p-1} + \delta\alpha_{p+1} + \kappa\alpha_p = 0, \qquad p = 0, \ldots, n \tag{32}$$

$(\alpha_{-1} \equiv 0 \equiv \alpha_{n+1})$, and in terms of the local components—as defined in the previous section—amount to the tensorial equations:

$$\frac{(-1)^{p-1}}{(p-1)!}\alpha_{[\mu_1 \ldots \mu_{p-1},\mu_p]} + \frac{(-1)^{p+1}}{p!}\alpha_{\mu_1 \ldots \mu_p \rho}{}^{;\rho} + \frac{\kappa}{p!}\alpha_{\mu_1 \ldots \mu_p} = 0,$$

$$p = 0, \ldots, n \tag{33}$$

If one would like to work in place of α_p's with the equivalent objects $\check{\alpha}_p := * \alpha_{n-p}$, $p = 0, \ldots, n$, then (32) and (33) can be shown to be equivalent to

$$(-1)^{n-p+1} d\check{\alpha}_{p-1} + (-1)^{n-p}\delta\check{\alpha}_{p+1} + \kappa\check{\alpha}_p = 0$$

and

$$\frac{(-1)^n}{(p-1)!}\check{\alpha}_{[\mu_1 \ldots \mu_{p-1},\mu_p]} + \frac{(-1)^{n+1}}{p!}\check{\alpha}_{\mu_1 \ldots \mu_p \rho}{}^{;\rho} + \frac{\kappa}{p!}\check{\alpha}_{\mu_1 \ldots \mu_p} = 0$$

This explicitly exhibits the fact that $\alpha_p \Leftrightarrow \check{\alpha}_p$ *is not* a symmetry of the studied equations.

These differential equations for $\alpha_{\mu_1 \ldots \mu_p}$, $p = 0, \ldots, n$ which involve the metric $g_{\mu\nu}$, possess then a simple action principle. Indeed, with $\mathscr{D} \in M$, and

$(\alpha, \beta)_{\mathscr{D}}$ defined by (26), consider a functional of $\alpha_{\mu_1 \ldots \mu_p}$, $p = 0, \ldots, n$ and $g_{\mu\nu}$ defined by

$$A := \tfrac{1}{2}(d\alpha + \delta\alpha + \kappa\alpha, \alpha)_{\mathscr{D}} \tag{34}$$

The so-defined action A, bilinear in $\alpha = \oplus_{p=0}^{n} \alpha_p$ —stated more explicitly—amounts to

$$\begin{aligned} A &= \frac{1}{2} \int_{\mathscr{D}} \mathscr{I}[d\alpha + \delta\alpha + \kappa\alpha, \alpha] \\ &= \frac{1}{2} \int_{\mathscr{D}} * \left\{ \sum_{p=0}^{n} (d\alpha_{p-1} + \delta\alpha_{p+1} + \kappa\alpha_p) \lrcorner \alpha_p \right\} \\ &= \frac{1}{2} \int_{\mathscr{D}} \left\{ \sum_{p=0}^{n} (d\alpha_{p-1} + \delta\alpha_{p+1} + \kappa\alpha_p) \lrcorner \alpha_p \right\} \sqrt{/}\, dx^1 \wedge \cdots \wedge dx^n \\ &=: \int_{\mathscr{D}} L\, dx^1 \wedge \cdots \wedge dx^n \end{aligned} \tag{35}$$

The second line of (35) provides the explicit definition of A in the chart independent language, the third is meant—assuming that the chart $\{x^\mu\}$ covers $\mathscr{D}$—to provide the definition of the Lagrangian L from the fourth line, L being understood in the conventional sense. The first line with $\mathscr{I}[\alpha, \beta]$ defined in (24) is the most convenient in showing that A understood as a functional of α and g, $A = A[\alpha, g]$, when varied with respect to α, leads to our dynamical equations $(d + \delta + \kappa)\alpha = 0$.

■ Indeed, because A is bilinear in α, denoting by $h = \oplus_{p=0}^{n} h_p$ the variation of α, we have

$$\begin{aligned} A[\alpha + h, g] = A[\alpha, g] &+ \frac{1}{2} \int_{\mathscr{D}} \mathscr{I}[d\alpha + \delta\alpha + \kappa\alpha, h] \\ &+ \frac{1}{2} \int_{\mathscr{D}} \mathscr{I}[dh + \delta h + \kappa h, \alpha] + A[h, g] \end{aligned} \tag{36}$$

On the other hand, remembering $\mathscr{I}[\alpha, \beta] = \mathscr{I}[\beta, \alpha]$ and the crucial property of $\mathscr{I}$, (25), we have

$$\begin{aligned} \mathscr{I}[dh + \delta h + \kappa h, \alpha] = \mathscr{I}&[d\alpha + \delta\alpha + \kappa\alpha, h] \\ &+ d\left\{ \sum_{p=0}^{n} (h_p \wedge * \alpha_{p+1} - \alpha_p \wedge * h_{p+1}) \right\} \end{aligned} \tag{37}$$

and consequently, applying Gauss–Stokes–Ostrogracki theorem, we have

$$A[\alpha+h,g]=A[\alpha,g]+\int_{\mathscr{D}}\mathscr{J}[d\alpha+\delta\alpha+\kappa\alpha,h]$$

$$+\int_{\partial\mathscr{D}}\sum_{p=0}^{n}(h_p\wedge *\alpha_{p+1}-\alpha_p\wedge *h_{p+1})+A[h,g] \quad (38)$$

where $\partial\mathscr{D}$ is the boundary of $\mathscr{D}\subset M$. Therefore, if according to the standard ideology of the action principles, the first variation of A is supposed to vanish for an arbitrary variation of α ($=h$), constrained to vanishing on $\partial\mathscr{D}$, for *every* $\mathscr{D}$, it then follows by employing duBois–Reymond lemma, that $(d+\delta+\kappa)\alpha=0$, $\Rightarrow$ (32) equations, or their local tensorial form (33). ■ ■

We should like to observe that because of (27) the two "kinetic" terms of our action, i.e., the terms involving d and δ operations are equivalent modulo a surface term. Nevertheless, we have chosen to work with the action A in its present form, because it shares with the standard action for the Dirac equations one important property: it vanishes along the integral variety.

The local Lagrangian L which corresponds to our action is given explicitly in its tensorial form according to

$$L:=\frac{1}{2}\sqrt{/}\left\{\sum_{p=1}^{n}\frac{(-1)^{p-1}}{(p-1)!}\alpha_{\mu_1\cdots\mu_{p-1},\mu_p}\alpha^{\mu_1\cdots\mu_p}\right.$$

$$\left.+\sum_{p=0}^{n-1}\frac{(-1)^{p+1}}{p!}\alpha_{\mu_1\cdots\mu_p\rho}{}^{;\rho}\alpha^{\mu_1\cdots\mu_p}+\sum_{p=0}^{n}\frac{\kappa}{p!}\alpha_{\mu_1\cdots\mu_p}\alpha^{\mu_1\cdots\mu_p}\right\} \quad (39)$$

Now, our action A is invariant with respect to the general coordinate transformations. Therefore, it follows according to the well-known theorem, that, modulo our field equations $(d+\delta+\kappa)\alpha=0$, the symmetric tensor

$$\sqrt{/}\,\Upsilon_{\mu\nu}:=2\frac{\delta}{\delta g^{\mu\nu}}A \quad (40)$$

is "covariantly conservative":

$$\Upsilon_{\mu\nu}{}^{;\nu}=0 \quad (41)$$

In the case of the real structure (M,g) with $n=4$ and the signature

$(+,-,-,-)$, $\Upsilon_{\mu\nu}$ has the interpretation of the energy-momentum tensor. From (40) it can be worked out that this tensor amounts to

$$\Upsilon_{\mu\nu} := \left\{ \sum_{p=1}^{n} \frac{(-1)^{p-1}}{(p-1)!} p\alpha_{[\mu_1 \dots \mu_{p-1},(\mu]} - \sum_{p=1}^{n-1} \frac{(-1)^{p+1}}{(p-1)!} \alpha_{\mu_1 \dots \mu_{p-1}(\mu|\rho|}{}^{;\rho} \right\} \alpha^{\mu_1 \dots \mu_{p-1}}{}_{\nu)}$$

$$- \frac{1}{2} g_{\mu\nu} \left\{ \sum_{p=1}^{n} \frac{(-1)^{p-1}}{(p-1)!} \alpha_{\mu_1 \dots \mu_p, \mu_p} - \sum_{p=0}^{n-1} \frac{(-1)^{p+1}}{p!} \alpha_{\mu_1 \dots \mu_p \rho}{}^{;\rho} \right\} \alpha^{\mu_1 \dots \mu_p} \tag{42}$$

[We recall that $[\dots]$ denotes antisymmetrization, $(\dots)$ the symmetrization of a set of indices; an index which does not participate in these operations, like ρ in (42), is denoted $|\rho|$.]

The symmetric tensor $\Upsilon_{\mu\nu} = \Upsilon_{(\mu\nu)}$ defined above, bilinear in fields $\alpha_{\mu_1 \dots \mu_p}$ and linear in their derivatives, which does not contain any term proportional to κ, is somewhat analogous to the energy-momentum tensor for the Dirac spinor $\psi^+ \gamma_{(\mu} \overleftrightarrow{\nabla}_{\nu)} \psi$. With (42) understood as the definition of $\Upsilon_{\mu\nu}$, it is a nice exercise in the traditional tensor calculus to show that, assumed (33) equations, $\Upsilon_{\mu\nu}{}^{;\nu} = 0$ indeed follows. The Dirac equations imply however also the covariant conservation of the probability current, $j^{\mu}_{;\mu} = 0$, $j^{\mu} := \psi^+ \gamma^{\mu} \psi$ being algebraically constructed in terms of ψ, meaning by γ^{μ} the γ matrices in a curved space-time. The question arises: does an analogous covariant conservation law exist for the considered bosonic structure with $(d + \delta + \kappa) = \alpha = 0$? It turns out that the answer is positive: define a tensor bilinear in the components of $\alpha = \oplus_{p=0} \alpha_p$ and algebraically constructed from these:

$$P_{\mu\nu} := 2 \sum_{p=0}^{n-1} \frac{(-1)^p}{p!} \left(\alpha^{\mu_1 \dots \mu_p}{}_{\mu} \alpha_{\mu_1 \dots \mu_p \nu} + \alpha^{\mu_1 \dots \mu_p} \alpha_{\mu_1 \dots \mu_p \mu\nu} \right)$$

$$+ g_{\mu\nu} \sum_{p=0}^{n} \frac{(-1)^p}{p!} \alpha^{\mu_1 \dots \mu_p} \alpha_{\mu_1 \dots \mu_p} \tag{43}$$

Then one can show that modulo our (33) equations

$$P_{\mu\nu}{}^{;\nu} = 0 \tag{44}$$

■ Perhaps the simplest manner of proving this, consists in constructing (33) equations with $\alpha^{\mu_1 \cdots \mu_p \mu}$, and subtracting from the so-derived equality the same equations for $p \to p+1$, with $\mu_1 \ldots \mu_{p+1} \to \mu_1 \ldots \mu_p \mu$ constructed with $\alpha^{\mu_1 \cdots \mu_p}$, so that the κ terms cancel out. By taking then the sum $\Sigma_{p=0}^{n}$ of the so-obtained equations with the free index μ, one easily sees that it indeed amounts to $P_{\mu\nu}{}^{;\nu} = 0$, with $P_{\mu\nu}$ defined in (43). ■ ■

Further on, we shall see that in the case of a real (M, g) with hyperbolic signature P_{tt} is positive definite parallelling the corresponding property of the temporal component of the Dirac current.

There exist still other covariant conservation laws which accompany our dynamical equations $(d + \delta + \kappa)\alpha = 0$. In particular, our equations imply the Klein–Gordon equations $(\Delta - \kappa^2)\alpha_p = 0$, $p = 0, \ldots, n$ each of these possessing an autonomous action principle with the corresponding action integral given by

$$A_p = A_p[\alpha_p] := \frac{1}{2} \int_{\mathscr{D}} * \left\{ d\alpha_p \lrcorner d\alpha_p + \delta\alpha_p \lrcorner \delta\alpha_p - \kappa^2 \alpha_p \lrcorner \alpha_p \right\} \tag{45}$$

with $\mathscr{D} \subset M$ having the boundary $\partial\mathscr{D}$. ■ Indeed, by employing the general rule:

$$*\left(d\alpha_p \lrcorner \beta_{p+1}\right) - *\left(\alpha_p \lrcorner \delta\beta_{p+1}\right) = d\left(\alpha_p \wedge * \beta_{p+1}\right) \tag{46}$$

valid for any α_p, β_{p+1} [this is an equivalent form of (25)], one easily sees that

$$\begin{aligned} A_p\left[\alpha_p + h_p\right] = A_p[\alpha_p] + \int_{\mathscr{D}} * h_p \lrcorner \left(\Delta\alpha_p - \kappa^2 \alpha_p\right) \\ + \int_{\partial\mathscr{D}} \left(h_p \wedge * d\alpha_p - \delta\alpha_p \wedge * h_p\right) + A_p\left[h_p\right] \end{aligned} \tag{47}$$

Therefore, with "small" h_p (understood as variation of α_p) vanishing on $\partial\mathscr{D}$, the vanishing of the first variation of A_p for every $\mathscr{D}$ implies $(\Delta - \kappa^2)\alpha_p = 0$. ■ ■

But A_p is an invariant with respect to arbitrary coordinate transformations. Therefore all n of the symmetric tensors:

$$\sqrt{/}\, \Upsilon_{(p)\mu\nu} := \frac{\delta A_p}{\delta g^{\mu\nu}}, \qquad p = 0, \ldots, n \tag{48}$$

are covariantly conservative

$$\Upsilon_{(p)\mu\nu}{}^{;\nu} = 0 \tag{49}$$

modulo $(\Delta - \kappa^2)\alpha_p = 0$ equations.

It is perhaps of interest to notice that the action integral (45) with the $\delta\alpha_p \lrcorner \delta\alpha_p$ term *missing*,

$$A'_p := \frac{1}{2}\int_{\mathscr{D}} *\left\{ d\alpha_p \lrcorner d\alpha_p - \kappa^2\alpha_p \lrcorner \alpha_p \right\} \tag{50}$$

varied with respect to α_p leads to

$$(\delta d - \kappa^2)\alpha_p = 0 \tag{51}$$

If κ is $\neq 0$, by acting on this equation with δ one infers that $\delta\alpha_p = 0$, and consequently (51) is equivalent to

$$(\Delta - \kappa^2)\alpha_p = 0, \qquad \delta\alpha_p = 0 \tag{52}$$

Note that the dynamical scheme described above amounts to an invariant generalization for the case of an arbitrary (M, g) and α_p with p arbitrary, of the Proca treatment of a massive vector meson with the Lorentz condition.

4. THE $(d + \delta + \kappa)\alpha = 0$ EQUATIONS: SPECIAL CASES AND THEIR INTERPRETATION

Consider the studied differential structure in the case of (M, g) real with the hyperbolic signature for the "realistic" subcase of $n = 4$, signature $(+, -, -, -)$, and parallely, for the sub case of $n = 2$, signature $(+, -)$, which can be considered as a model of the realistic situation.[3]

For both cases $n = 4, 2$ according to (19) and (23) we have

$$** \Lambda^p = \begin{cases} \Lambda^p, & p \text{ odd}, \\ -\Lambda^p, & p \text{ even}, \end{cases} \qquad \delta = *d* \tag{53}$$

[3]A suggestion by Marcos Moshinsky to investigate this simplest possible model which admits a dynamical interpretation is gratefully appreciated.

and the (14) equations amount correspondingly to

$$n=2 \rightarrow \begin{cases} 0+\delta\alpha_1+\kappa\alpha_0=0 \\ d\alpha_0+\delta\alpha_2+\kappa\alpha_1=0\,, \\ d\alpha_1+0+\kappa\alpha_2=0 \end{cases} \qquad n=4 \rightarrow \begin{cases} 0+\delta\alpha_1+\kappa\alpha_0=0 \\ d\alpha_0+\delta\alpha_2+\kappa\alpha_1=0 \\ d\alpha_1+\delta\alpha_3+\kappa\alpha_2=0 \\ d\alpha_2+\delta\alpha_4+\kappa\alpha_3=0 \\ d\alpha_3+0+\kappa\alpha_4=0 \end{cases} \tag{54}$$

If we replace for $n=2$, α_2 by $-*\check{\alpha}_0$, and for $n=4$, α_4 by $-*\check{\alpha}_0$ and α_3 by $*\check{\alpha}_1$, the forms with "ˇ" above corresponding to pseudotensors, then our equations assume the equivalent form of

$$n=2 \rightarrow \begin{cases} \delta\alpha_1+\kappa\alpha_0=0 \\ d\alpha_0+*\,d\check{\alpha}_0+\kappa\alpha_1=0\,, \\ *\,d\alpha_1+\kappa\check{\alpha}_0=0 \end{cases} \qquad n=4 \rightarrow \begin{cases} \delta\alpha_1+\kappa\alpha_0=0 \\ d\alpha_0+\delta\alpha_2+\kappa\alpha_1=0 \\ d\alpha_1+*\,d\check{\alpha}_1+\kappa\alpha_2=0 \\ d\check{\alpha}_0+*\,d\alpha_2+\kappa\check{\alpha}_1=0 \\ \delta\check{\alpha}_1+\kappa\check{\alpha}_0=0 \end{cases} \tag{55}$$

Notice that because $*$ commutes with Δ [this is true for an arbitrary (M,g) as a consequence of (55)]

$$n=2 \rightarrow (\Delta-\kappa^2)\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \check{\alpha}_0 \end{pmatrix}=0, \qquad n=4 \rightarrow (\Delta-\kappa^2)\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \check{\alpha}_1 \\ \check{\alpha}_0 \end{pmatrix}=0 \tag{56}$$

In terms of chart components the objects involved in the present statement of our equations amount of course to

$$n=2 \rightarrow \begin{pmatrix} \alpha_0 \\ \alpha_\mu \\ \check{\alpha}_0 \end{pmatrix}, \qquad n=4 \rightarrow \begin{pmatrix} \alpha_0 \\ \alpha_\mu \\ \alpha_{\mu\nu} \\ \check{\alpha}_\mu \\ \check{\alpha}_0 \end{pmatrix} \tag{57}$$

Denote now by D_2, D_4 the blocks of equations (55) for $n = 2,4$. Then, if $\kappa \neq 0$, there hold interesting equivalences:

$$n = 2 \rightarrow \left.\begin{aligned} (\Delta - \kappa^2)\alpha_1 = 0 \\ \alpha_0 + i\check{\alpha}_0 \coloneqq -\kappa^{-1}(\delta + i * d)\alpha_1 \end{aligned}\right\} \Leftrightarrow D_2 \Leftrightarrow \left\{\begin{aligned} &(\Delta - \kappa^2)(\alpha_0 + i\check{\alpha}_0) = 0 \\ &\alpha_1 \coloneqq -\kappa^{-1}(d\alpha_0 + * d\check{\alpha}_0) \end{aligned}\right. \tag{58}$$

and

$$n = 4 \rightarrow \left.\begin{aligned} (\Delta - \kappa^2)(\alpha_1 + i\check{\alpha}_1) = 0 \\ \alpha_0 + i\check{\alpha}_0 \coloneqq -\kappa^{-1}\delta(\alpha_1 + i\check{\alpha}_1) \\ \alpha_2 \coloneqq -\kappa^{-1}(d\alpha_1 + * d\check{\alpha}_1) \end{aligned}\right\}$$

$$\Leftrightarrow D_4 \Leftrightarrow \left\{\begin{aligned} &(\Delta - \kappa^2)\alpha_2 = 0 \\ &(\Delta - \kappa^2)(\alpha_0 + i\check{\alpha}_0) = 0 \\ &\alpha_1 + i\check{\alpha}_1 \coloneqq -\kappa^{-1}\left[d(\alpha_0 + i\check{\alpha}_0) + (\delta + i * d)\alpha_2\right] \end{aligned}\right. \tag{59}$$

By "$\coloneqq$" we mean here, of course, "equal from definition." The definitions in (58) and (59) correspond directly to some of D_2 and D_4 equations; the point is that these definitions used in the postulated simultaneously Klein–Gordon equations, by employing the properties of d, δ, and $*$, are exactly sufficient to show that K. G. equations are equivalent to the "missing" equations of D_2 and D_4 correspondingly.

On the first sight, the equivalences established above seem to "trivialize" the studied dynamical structure: D_4 is just equivalent to the usual theory of a complex vector boson, $\alpha_1 + i\check{\alpha}_1 = (\alpha_\mu + i\check{\alpha}_\mu)\,dx^\mu$ submitted to the Klein–Gordon equation, brought to the shape of the first differential order equations, by appropriate definitions for some combinations of its first derivatives. Equivalently, it can be just considered as the usual theory of the real vector particles of spins 1^+ and 1^-, endowed with the same mass. However, D_4 understood as equivalent to the right-hand side block in (59) can be interpreted in a dual manner: it corresponds to the usual theory of a complex scalar boson $\alpha_0 + i\check{\alpha}_0$—equivalently real scalar bosons of spins 0^+ and 0^-—and the real boson of spin 1, $\alpha_{\mu\nu}$, which thus corresponds to the representation $D(1,0)\times D(0,1)$, all these bosons being endowed with the same mass. We argue that the existence of this dual "usual" interpretation of D_4, after all, exhibits the nontrivial nature of our differential structure: bosons with spin $1^\pm$ (representations $D(1/2,1/2)^\pm$) are *equivalent* to bosons $0^\pm$ (representations $D(0,0)^\pm$) and a $D(1,0)\times D(0,1)$ boson, spin $s=1$. Question arises: which of the dual interpretations ought to be assumed as physically valid? A reasonable answer appears to be that simultaneously *both are valid*, with the "middle of the road" first differential order equations D_4 understood as the dynamical equations. It still may appear strange that a theory of spin-$1^\pm$ particles results *equivalent* to the theory of spin 1, representation $D(1,0)\times D(0,1)$ particles *and* $0^\pm$ particles. At this point, it can be observed that in the attempts to construct a wave equation for bosons, usually spin-1 and spin-0 particles appear simultaneously, like, e.g., within the Duffin–Kemmer theory. Moreover, notice that if we specialize the equivalences (59) for the case of the generalized Proca scheme discussed at the end of the previous section with the "Lorentz conditions" imposed for the vector particles, then these equivalences reduce to

$$n=4\rightarrow \left.\begin{aligned}(\Delta-\kappa^2)(\alpha_1+i\breve{\alpha}_1)&=0\\ \delta(\alpha_1+i\breve{\alpha}_1)&=0\\ \alpha_2:=-\kappa^{-1}(d\alpha_1+d\breve{\alpha}_1)&=0\end{aligned}\right\}\Leftrightarrow\left\{\begin{aligned}\delta\alpha_2+\kappa\alpha_1&=0\\ d\alpha_1+*\,d\breve{\alpha}_1+\kappa\alpha_2&=0\\ *\,d\alpha_2+\kappa\breve{\alpha}_1&=0\end{aligned}\right\}$$

$$\Leftrightarrow\begin{cases}(\Delta-\kappa^2)\alpha_2=0\\ \alpha_1+i\breve{\alpha}_1:=-\kappa^{-1}(\delta+i*d)\alpha_2\end{cases}\tag{60}$$

the left and right block—as well as the "middle of the road" equations—involving only the objects with the spin $s=1$ [although belonging to different representations: the "Lorentz condition" $\delta(\alpha_1+i\breve{\alpha}_1)=0$ eliminates the "longitudinal" scalar particles α_0 and $\breve{\alpha}_0$].

The comment above valid for $n=4$ certainly reinforces the well-known virtues (positive definiteness of energy, etc.) of the Proca scheme with its Lorentz condition.

Now, the equivalence (58) for the case $n=2$ has a similar interpretation: a complex "scalar particle" $\alpha_0+i\breve{\alpha}_0$, submitted to the Klein–Gordon equation turns out to be equivalent—in the sense of the usual theory—to a real "vector particle." Here no trick "*à la* Proca" can assure the same "spin" of the left and right blocks of (58). For this reason, further on, developing the interpretation of $(d+\delta+\kappa)\alpha=0$ in various cases, we shall not try prematurely to imitate the Proca scheme for $n=4$, (equivalent in terms of original α_p's to the assumption that $\alpha_0=0=\alpha_4$, $\rightarrow\delta\alpha_1=0=d\alpha_3$, via the D_4 equations with $\kappa\neq0$). Notice, however, that our general action (35) specialized for the case: $n=4$, signature $(+,-,-,-)$—with (53) valid

—if still constrained, "by decree," by the condition $\alpha_0 = 0 = \alpha_4$, amounts to

$$A' = \frac{1}{2}\int_{\mathscr{D}} * \{(\delta\alpha_2 + \kappa\alpha_1)\lrcorner\alpha_1 + (d\alpha_1 + \delta\alpha_3 + \kappa\alpha_2)\lrcorner\alpha_2 + (d\alpha_2 + \kappa\alpha_3)\lrcorner\alpha_3\} \tag{61}$$

and with $\alpha_3 = *\check{\alpha}_1$ it reduces to

$$A' = \frac{1}{2}\int_{\mathscr{D}} * \{(\delta\alpha_2 + \kappa\alpha_1)\lrcorner\alpha_1 + (d\alpha_1 + *\,d\check{\alpha}_1 + \kappa\alpha_2)\lrcorner\alpha_2 - (*\,d\alpha_2 + \kappa\check{\alpha}_1)\lrcorner\check{\alpha}_1\} \tag{62}$$

[Showing this, we used the identity valid under the present assumptions, $*\,\alpha_1 \lrcorner * \beta_1 = -\,\alpha_1 \lrcorner \beta_1$, which follows from (21).] This action when varied with respect to $\alpha_1, \alpha_2, \check{\alpha}_1$ is easily seen to lead to the "middle of the road" equations from (60) and vanishes along the integral variety of these equations. Then, acting with δ on the first and the third of these equations, we easily infer that if $\kappa \neq 0$, they imply the "Lorentz condition":

$$\delta(\alpha_1 + i\check{\alpha}_1) = 0 \tag{63}$$

Obviously, this generalizes the "Proca process" discussed at the end of the previous section, applicable for the second-order differential equations $(\Delta - \kappa^2)\alpha_r = 0$, to the case of our first-order differential equations with $n = 4$ and signature $(+,-,-,-)$.

It should be perhaps observed at this point that the results of the first part of this section were technically obtained by using the algebraic properties of $*$, $\lrcorner$, d, and δ. To demonstrate them working in terms of local chart components would involve some rather messy computations.

In conventional physical theories treating a particle of the given spin s one usually commits the field which corresponds to that particle to be the carrier of a *fixed* representation $D(k,l)$, $k + l = s$, having thus in principle $2s+1$ options for this fixed choice. Equivalences (60) indicate that there are possible dynamical theories where although the spin is fixed, the different representations compatible with it are treated on equal footing. [In our specific case $D(1/2,1/2)^{\pm}$ real fields are equivalently described by a real $D(1,0)\times\overline{D(0,1)}$ field.]

In the second part of this section we would like to study $(d + \delta + \kappa)\alpha = 0$ equation in the n-dimensional hyperbolic case understood as the "evolution equation." We will do so assuming strongly that $V_n = V_1 \times V_{n-1}$, $M_n = \mathscr{R}\times$

M_{n-1} and

$$g_n = g_1 - g_{n-1}, \qquad g_1 = dt \underset{s}{\otimes} dt, \qquad g_{n-1} = g_{ab}\, dx^a \underset{s}{\otimes} dx^b$$

$$a, b, \ldots = 1, \ldots, n-1, \qquad \partial_t g_{ab} = 0, \qquad g_{ab} \text{ positive definite} \quad (64)$$

This is of course a much stronger assumption than the existence of a temporal killing vector $\partial_t \sim t^\mu$ for a hyperbolic V_n of signature $(+, -1, \ldots, -1)$. With $V_n = V_1 \times V_{n-1}$, t_μ is covariantly constant, $t_{\mu;\nu} = 0$. On the other hand, if $V_1 \times V_{n-1}$ is flat, then V_{n-1} is flat, and there exist the privileged Cartesian coordinates $\{x^a\}$ such that $g_{ab} = \delta_{ab}$. Thus, studying the case of $V_1 \times V_{n-1}$ we implicitly also cover the physically interesting case of flat g_n with the "Poincaré group" $\mathscr{R}^n$ ⓢ $O(1, n-1)$.

With $V_1 \times V_{n-1}$ described in chart $\{x^\mu\} = \{t, x^a\}$ the independent components of $\alpha_{\mu_1 \ldots \mu_p}$, $p = 0, \ldots, n$, amount to the independent components of

$$\alpha_{a_1 \ldots a_p} \quad \text{and } \alpha_{a_1 \ldots a_p t}, \qquad p = 0, \ldots, n-1 \quad (65)$$

while these objects can be interpreted as the-skew tensors in the sense of V_{n-1}, depending *parametrically* on t. We observe then that by specializing (33) equations for $\mu_1 \ldots \mu_p \to a_1 \ldots a_p$, and then with $p \to p+1$ for $\mu_1 \ldots \mu_{p+1} \to a_1 \ldots a_p t$ under the present assumptions concerning $g_{\mu\nu}$, after some work, one arrives at the conclusion that they are equivalent to

$$\partial_t \alpha_{a_1 \ldots a_p t} + p\alpha_{[a_1 \ldots a_{p-1}, a_p]} - \alpha_{a_1 \ldots a_p b}{}^{;b} + (-1)^p \kappa \alpha_{a_1 \ldots a_p} = 0$$

$$\partial_t \alpha_{a_1 \ldots a_p} - p\alpha_{[a_1 \ldots a_{p-1}|t|, a_p]} + \alpha_{a_1 \ldots a_p bt}{}^{;b} + (-1)^p \kappa \alpha_{a_1 \ldots a_p t} = 0 \quad (66)$$

The covariant derivatives "$(\ldots)^{;b}$" are meant here in the sense of the positive definite metric g_{ab} and of course these operations "ignore" the index t of $\alpha_{a_1 \ldots a_p t}$. The alternating signs in the first and second of (66) equations suggest that they can be spelled out as a single complex equation. Indeed, define a complex V_{n-1} tensor—parametrically dependent on t—

$$\beta_{a_1 \ldots a_p} := i^p \alpha_{a_1 \ldots a_p} + i^{p+1} \alpha_{a_1 \ldots a_p t}, \qquad p = 0, \ldots, n-1 \quad (67)$$

which encodes conveniently for our purposes all $2n$ of the independent components of $\alpha_{\mu_1 \ldots \mu_p}$, $p = 0, \ldots, n$. Then, multiplying (66) equations by i^{p+1} and i^p and adding we infer that they are equivalent to the complex evolution equation

$$[\partial_t - (-1)^p i\kappa]\beta_{a_1\ldots a_p} - p\beta_{[a_1\ldots a_{p-1},a_p]} - \beta_{a_1\ldots a_p b}{}^{;b} = 0, \qquad p = 0,\ldots,n-1 \tag{68}$$

These equations can be now stated invariantly with respect to the $V_{n-1} = (M_{n-1}, g_{n-1})$ structure. Indeed, define the complex valued p forms:

$$\beta_p := \frac{1}{p!}\beta_{a_1\ldots a_p}dx^{a_1}\wedge\cdots\wedge dx^{a_p}, \qquad p = 0,\ldots,n-1 \tag{69}$$

which can be understood as belonging to the *complexified* $\Lambda(M_{n-1})$, $\beta = \oplus_{p=0}^{n-1}\beta_p \in \Lambda(M_{n-1})$, endowed with the operations $(.\mathscr{C}, +, \wedge)$ depending *parametrically* on the real t. Notice that from the point of view of $V_n = V_1 \times V_{n-1}$ with the distinguished dt, these forms can be understood as concisely defined by

$$\beta_p := (-i)^p\, dt \lrcorner (\alpha_p \wedge dt + i\alpha_{p+1})$$

$$\Leftrightarrow \alpha_p = \mathrm{Re}\{(-i)^p[\beta_p + \beta_{p-1}\wedge dt]\} \tag{70}$$

Assume now the convention that the operations $*$, $\lrcorner$, $d,\ldots$ defined in Section 2 all refer to $V_n = V_1 \times V_{n-1}$ structure, while these operations when understood in the sense of $V_{n-1} = (M_{n-1}, g_{n-1})$ with its positive definite metric are to be denoted $*', \Box', d',\ldots$. Then (68) contracted with $(1/p!)dx^{a_1}\wedge\cdots\wedge dx^{a_p}$ assumes the V_{n-1} invariant form

$$[(-1)^p\partial_t - i\kappa]\beta_p + d'\beta_{p-1} + \delta'\beta_{p+1} = 0, \qquad p = 0,\ldots,n-1 \tag{71}$$

Using this invariant complex form of the evolution equations one can prove that

$$\partial_t\left(\sum_{p=0}^{n-1}\bar{\beta}_p \lrcorner' \beta_p\right) + \delta'\left(\sum_{p=0}^{n-1}\bar{\beta}_p \lrcorner' \beta_{p+1} + \mathrm{C.C.}\right) = 0 \tag{72}$$

This amounts to a conservation law valid in arbitrary coordinates for g_{n-1},

$$\partial_t(\sqrt{/}\,\rho) + \partial_a(\sqrt{/}\,j^a) = 0 \tag{73}$$

where

$$\rho := \sum_{p=0}^{n-1} \bar{\beta}_p \lrcorner {}'\beta_p = \sum_{p=0}^{n-1} \frac{1}{p!} \bar{\beta}^{a_1 \dots a_p} \beta_{a_1 \dots a_p} \geqslant 0$$

$$- j_a\, dx^a := \sum_{p=0}^{n-1} \bar{\beta}_p \lrcorner {}'\beta_{p+1} + \text{C.C.},$$

$$\Rightarrow - j_a = \sum_{p=0}^{n-1} \frac{1}{p!} \bar{\beta}^{a_1 \dots a_p} \beta_{a_1 \dots a_p a} + \text{C.C.} \tag{74}$$

Notice that under present assumptions $\sqrt{/} = \sqrt{/}'$, $\sqrt{/}$ being defined in (36). In the formulas above indices are manipulated by the positive definite metric g_{ab}. The conservation law (72) written in the form of $j^{\mu}{}_{;\mu} = 0$, $j^{\mu} = (\rho, j^m)$ can be then according to the general results of the previous section seen as a consequence of the covariant $P_{\mu\nu}{}^{;\nu} = 0$ with $P_{\mu\nu}$ defined in (43): j^{μ} coincides with $j^{\mu} := t^{\nu} P_{\nu}^{\mu}$, where t^{ν} is the cotangent form of the covariantly constant ($t_{\mu;\nu} = 0$) Killing vector ∂_t. Observe that in terms of the objects (65) equivalent to $\alpha_{\mu_1 \dots \mu_p}$, $p = 0, \dots, n$, ρ and j_a are explicitly given as

$$\begin{aligned} \rho &= \sum_{p=0}^{n-1} \frac{1}{p!} \left(\alpha^{a_1 \dots a_p t} \alpha_{a_1 \dots a_p t} + \alpha^{a_1 \dots a_p} \alpha_{a_1 \dots a_p} \right) \geqslant 0 \\ - j_a &= \sum_{p=0}^{n-1} \frac{1}{p!} \left(\alpha^{a_1 \dots a_p t} \alpha_{a_1 \dots a_p a} - \alpha^{a_1 \dots a_p} \alpha_{a_1 \dots a_p a t} \right) \end{aligned} \tag{75}$$

the "spatial" indices being manipulated by the positive definite g_{ab}.

We observe now that in the studied case of $V_n = V_1 \times V_{n-1}$, with the "time" t distinguished and "frozen" (i.e., free only modulo $t \to t + t_0$, $t_0 = \text{const}$), $\Lambda(M_n) = \Lambda(\mathscr{R} \times M_{n-1})$ has the natural structure of a Hilbert space over $\mathscr{R}$. Indeed, working with the local components of $\alpha = \oplus_{p=0}^{n} \alpha_p \in \Lambda(M_n)$ described by V_{n-1} tensorial objects (65) dependent parametrically on t, we can define the real symmetric scalar product on $\Lambda(M_n)$ as given by

$$\begin{aligned} \left\langle \alpha'(t'), \alpha''(t'') \right\rangle := \int_{M_{n-1}} & \sqrt{/}'\, dx^1 \wedge \cdots \wedge dx^{n-1} \\ & \times \sum_{p=0}^{n-1} \frac{1}{p!} \Big\{ \alpha'^{a_1 \dots a_p}(t') \alpha''_{a_1 \dots a_p}(t'') \\ & \qquad + \alpha'^{a_1 \dots a_p t'}(t') \alpha''_{a_1 \dots a_p t''}(t'') \Big\} \end{aligned} \tag{76}$$

indices being raised by the positive definite g_{ab}. This construction is invariant from the point of view of V_{n-1}: indeed, (70), equivalently (67), establish clearly an isomorphism $\Lambda(M_n) \ni \alpha = \alpha(t) \Leftrightarrow \beta \in$ complexified $\Lambda(M_{n-1})$, and with $\alpha'(t') \Leftrightarrow \beta'$, $\alpha''(t'') \Leftrightarrow \beta''$, (76) can be equivalently stated as

$$\langle \alpha'(t'), \alpha''(t'') \rangle = \mathrm{Re}(\overline{\beta'}, \beta'')$$

$$(\overline{\beta'}, \beta'') := \int_{M_{n-1}} *' \sum_{p=0}^{n-1} \overline{\beta'_p} \lrcorner' \beta''_p$$

$$= \int_{M_{n-1}} \left(\sum_{p=0}^{n-1} \frac{1}{p!} \overline{\beta'}^{a_1 \ldots a_p} \beta''_{a_1 \ldots a_p} \right) \sqrt{\gamma}' \, dx^1 \wedge \cdots \wedge dx^{n-1} \tag{77}$$

Notice that with this interpretation, $\langle \alpha'(t'), \alpha''(t'') \rangle = \langle \alpha''(t''), \alpha'(t') \rangle$ and $\langle \alpha(t), \alpha(t) \rangle = 0 \Leftrightarrow \alpha(t) = 0$, are self-evident properties of our $\langle , \rangle$. The convergence of the integrals with M_{n-1} compact is trivial, and when, e.g., $M_{n-1} = \mathscr{R}^{n-1}$ to assure it, we simply constrain $\alpha_{a_1 \ldots a_p}(t)$ and $\alpha_{a_1 \ldots a_p t}(t)$ V_{n-1} tensors to be "square integrable" (L^2 scalar product) demanding that $(\overline{\beta}, \beta) < \infty$.[4])

A general remark: the temporal evolution of states $|t\rangle$" must allow their linear superposition for different times, i.e., with $z', z'' \in \mathfrak{F}$, $z'|t'\rangle + z''|t''\rangle$, if the derivative $\partial_t |t\rangle$ is supposed to make sense; but independently from the specific choice for the field $\mathfrak{F}$, $z'|t'\rangle + z''|t''\rangle$ is then *not to be* interpreted as a state corresponding to a definite time t! On the other hand, *different* states at the same t, $|t\rangle$' canonically understood as "the maximum information about the physical system at t"[Dirac, 1957], when superposed, must be understood as a state at t, this superposition process constrained to

[4]It is very tempting to consider complexified $\Lambda(M_{n-1})$, with $\beta \in \Lambda(M_{n-1})$ depending parametrically on t, as a natural Hilbert space structure *over* $\mathscr{C}$, with the scalar product $(\overline{\beta}', \beta'')$: this would open doors to the standard quantum mechanical interpretation of the studied dynamical structure. We resist this temptation, because assumed this interpretation the superposition of "states" with $z', z'' \in \mathscr{C}$, $\beta = z'\beta' + z''\beta''$, is not consistent with the V_n invariant nature of the isomorphism $\Lambda(M_n) \ni \alpha(t) \Leftrightarrow \beta \in$ complexified $\Lambda(M_{n-1})$: assuming such a superposition of the components of $\alpha(t)$ it would lead to the general linear combinations of $\alpha'_{a_1 \ldots a_p}(t'), \alpha''_{a_1 \ldots a_p}(t'')$ with $\alpha_{a_1 \ldots a_p t'}(t')$, and $\alpha''_{a_1 \ldots a_p t''}(t'')$. Hence, even with β's taken at the equal times, $t' = t = t''$, $\alpha(t)$ read off from $\alpha(t) \Leftrightarrow \beta = z'\beta' + z''\beta''$ according to (67) *would not* lead in general to the V_n covariant objects $\alpha_{\mu_1 \ldots \mu_p}$, $p = 0, \ldots, n$. With z', z'' constrained to $\mathscr{R}$ only in the main text, this difficulty disappears.

be compatible with the covariance requirements of the theory. The construction outlined in (76) permits us to interpret consistently $\alpha(t) \Leftrightarrow \beta$ as the "evolutionary states" from a Hilbert space over $\mathscr{R}$, in the sense discussed above.

From the point of view of our $\langle , \rangle$ product, the notion of the complex β ($\Leftrightarrow \alpha(t)$) somewhat loses its structural naturality, remaining only a convenient technical device in exhibiting the simple nature of the $(d + \delta + \kappa)\alpha = 0$ equation for $\alpha \in \Lambda(M_n)$ interpreted as the evolution equation for the case of a hyperbolic $V_n \times V_{n-1}$.

Now, assumed our evolution equation

$$\begin{aligned}\partial_t \langle \alpha(t), \alpha(t) \rangle &= \partial_t \mathrm{Re}(\bar{\beta}, \beta)\\ &= \partial_t \int_{M_{n-1}} \rho \sqrt{/}' dx^1 \wedge \cdots \wedge dx^{n-1}\\ &= [\text{via } (73)] = \int_{\partial M_{n-1}} \cdots = 0\end{aligned} \tag{78}$$

if either M_{n-1} is compact ($\partial M_{n-1} = 0$) or α is square integrable, $\langle \alpha, \alpha \rangle < \infty$. Therefore, our evolution equation is compatible with the normalization condition

$$\langle \alpha(t), \alpha(t) \rangle = 1 \tag{79}$$

assumed which, $\rho \geqslant 0$ can be interpreted as the probability of localizing the dynamical system in the invariant volume $*'.1 = \sqrt{/}' dx^1 \wedge \cdots \wedge dx^{n-1}$.

All this is consistent for the case of a hyperbolic $V_n = V_1 \times V_{n-1}$ with the time t understood as "frozen," and not allowed to be mixed up with coordinates of M_{n-1}. The question arises whether the proposed interpretation still applies when $V_1 \times V_{n-1}$ is flat and the metric given in cartesian coordinates

$$g_n = dt \underset{s}{\otimes} dt - \delta_{ab} dx^1 \underset{s}{\otimes} dx^b = \eta_{\mu\nu} dx^\mu \underset{s}{\otimes} dx^\nu \tag{80}$$

possesses the Poincaré group of symmetries, with the Lorentz transformations which mix t with x^a being allowed. Of course, in a fixed Lorentz frame (t "frozen"), the interpretation applies without changes. Because under a general Poincaré transformation $x^\mu = (t, x^a) \to x^{\mu'} = (t', x^{a'})$, with $\alpha_{\mu_1 \cdots \mu_p}$ transforming tensorially to $\alpha_{\mu'_1 \cdots \mu'_p}$, all objects involved in our construction are *uniquely* defined and fulfill the same form invariant equations, we can argue that this is precisely *enough* in order to impose again—from the point of view of the primed frame—the same probabilistic interpretation as before, but now with the "time slice" $t' = \text{const}$ distinguished in place of the previous t. A moment of reflection leads to conclusion that the stand assumed above is consistent. All that is needed is a *unique* prescription how the observers at *any* time slice perceive probabilistically the dynamics under study. This coincides essentially with the argument which allows the consistent probabilistic interpretation of the special relativistic Dirac equations (Dirac, 1957). However, while the Dirac probability current $j^\mu = \psi^+ \gamma^\mu \psi$ is a 4-vector and the normalization condition for the states $|\psi\rangle$ can be postulated in the form *independent* from the choice for a time slice,

$$\int_\sigma j^\mu d_3\sigma_\mu = 1$$

σ being a spacelike surface, in our case the situation is different. The probability density ρ coincides (in the studied flat case) with the component $P_{tt} \geqslant 0$ of the tensor $P_{\mu\nu}$ from (43) which satisfies the conservation law:

$$P_{\mu\nu}{}^{,\nu} = 0 \tag{81}$$

Consequently, assuming $\alpha(t)$ square integrable ($\int_{M_{n-1}} P_{tt}\, dx^1 \wedge \cdots \wedge dx^{n-1} < \infty$) we have

$$N_\mu := \int_{M_{n-1}} P_\mu{}^t dx^1 \wedge \cdots \wedge dx^{n-1} \to \partial_t N_\mu = 0 \to N_\mu = \text{const}, \tag{82}$$

and under the Poincaré group $\int_{M_{n-1}} \rho\, dx^1 \wedge \cdots \wedge dx^{n-1}$ transforms like N_t component of the constant vector N_μ. Hence the normalization condition (79) is not invariant with respect to the different choices for the time slices. Although this causes that the probability densities ρ and ρ' related to time slicings of Poincaré equivalent systems of reference (t, x^a) and $(t', x^{a'})$, *do not* transform like a temporal component of a V_n vector, nevertheless the probabilistic interpretation remains consistent.

5. FLAT HYPERBOLIC (M, g): SOME EXPLICIT FORMS OF THE EQUATIONS FOR $n = 2,3,4$

If hyperbolic $V_1 \times V_{n-1}$ is flat, with the chart of M_{n-1} so arranged that $g_{ab} = \delta_{ab}$, then $(d + \delta + \kappa)\alpha = 0$ stated in the complex form (68). reduce to the simple

$$[\partial_t - (-1)^p i\kappa]\, \beta_{a_1 \ldots a_p} - p\beta_{[a_1 \ldots a_{p-1}, a_p]} - \beta_{a_1 \ldots a_p b, b} = 0,$$
$$p = 0, \ldots, n-1 \tag{83}$$

the relation between $\alpha_{\mu_1\ldots\mu_p}$, $p=0,\ldots,n$ and $\beta_{a_1\ldots a_p}$, $p=0,\ldots,n-1$ being defined by (67). Although it is not self-evident from the structure of (83) equations, these being equivalent according to (33) to

$$\frac{(-1)^{p-1}}{(p-1)!}\alpha_{[\mu_1\ldots\mu_{p-1},\mu_p]}+\frac{(-1)^{p+1}}{p!}\alpha_{\mu_1\ldots\mu_p\rho}{}^{,\rho}+\frac{\kappa}{p!}\alpha_{\mu_1\ldots\mu_p}=0,$$
$$p=0,\ldots,n \quad (84)$$

describe a *Poincaré invariant* structure.

Equations (83) constitute a very convenient tool if one wishes to exhibit the evolution process implicit in our basic $(d+\delta+\kappa)\alpha=0$, in a form which imitates the Schrödinger (or Dirac) wave equations:

$$(i\hbar\partial_t - H)\psi = 0 \quad (85)$$

This will be separately discussed for the cases of $n=2,3,4$. In doing so, we assume the standard Pauli notation for the basis of complex 2×2 matrices:

$$[1,\sigma_1,\sigma_2,\sigma_3]=\left[\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}0&-i\\i&0\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right] \quad (86)$$

For $n=2\to n-1=1$ β's have the only nontrivial components:

$$\begin{aligned}\beta_0&=\alpha_0+i\alpha_t\\ \beta_1&=i\alpha_1-\alpha_{1t}\equiv i\alpha_1-\check{\alpha}_0\end{aligned} \quad (87)$$

the real basic fields understood as α_0, $\alpha_\mu=(\alpha_t,\alpha_1)$ and $\alpha_{\mu\nu}=-\varepsilon_{\mu\nu}\check{\alpha}_0$ ($\alpha_2=-*\check{\alpha}_0$!). In terms of β's (83) equations reduce to

$$(\partial_t-i\kappa)\beta_0-\partial_1\beta_1=0,\qquad (\partial_t+i\kappa)\beta_1-\partial_1\beta_0=0 \quad (88)$$

and therefore, remembering $\kappa=mc/\hbar$ and defining a column with the complex entries:

$$n=2\to\psi=\begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix} \quad (89)$$

our equations amount to (85) with

$$n=2\to H:=\{\sigma_1(-i\hbar\partial_1)+mc\sigma_3\} \quad (90)$$

Notice that the conservative ρ is given here by

$$n = 2 \rightarrow \rho = \alpha_0^2 + \alpha_t^2 + \alpha_1^2 + \check{\alpha}_0^2 \tag{91}$$

In the case of $n = 3$, a similar argument applies: not entering into details, we observe only that here there are only four components of complex β's: $\beta_0, \beta_a, \beta_{ab} =: \varepsilon_{ab}\check{\beta}_0$, $a, b, \ldots = 1, 2$ and arranging these independent complex components in the form of a column

$$n = 3 \rightarrow \psi = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \check{\beta}_0 \end{pmatrix} \tag{92}$$

one easily finds that (83) is equivalent to (85) with

$$n = 3 \rightarrow H$$
$$:= -\left\{ \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array} \right)(-i\hbar\partial_1) + \left(\begin{array}{c|c} 0 & \sigma_1 \\ \hline \sigma_1 & 0 \end{array} \right)(-i\hbar\partial_2) + \left(\begin{array}{c|c} \sigma_3 & 0 \\ \hline 0 & -\sigma_3 \end{array} \right) mc \right\} \tag{93}$$

Now, in the most interesting "realistic" case of $n = 4$ the nontrivial components of complex β's amount to

$$\begin{cases} \beta_0 =: \varphi \\ \beta_a =: E_a \\ \beta_{ab} =: \varepsilon_{abc}\check{\beta}_c \\ \beta_{abc} =: \varepsilon_{abc}\check{\varphi} \end{cases} \Leftrightarrow \begin{cases} \varphi = \alpha_0 + i\alpha_t \\ E_a = i\alpha_a - \alpha_{at} \\ \check{B}_a = -(1/2)\varepsilon_{abc}(\alpha_{bc} + i\alpha_{bct}) = -(1/2)\varepsilon_{abc}\alpha_{bc} - i\check{\alpha}_a \\ \check{\varphi}_a = (1/3!)\varepsilon_{abc}(-i\alpha_{abc} + \alpha_{abct}) = -i\check{\alpha}_t + \check{\alpha}_0 \end{cases} \tag{94}$$

The real bosonic fields involved in this construction are of course α_0, $\alpha_\mu = (\alpha_t, \alpha_m)$, $\alpha_{\mu\nu} = \alpha_{[\mu\nu]} = (\alpha_{mt}, \alpha_{mn})$, $\check{\alpha}_\mu = (\check{\alpha}_t, \check{\alpha}_m)$ and $\check{\alpha}_0$ [$\alpha_3 = *\check{\alpha}_1$, $\alpha_4 = -*\check{\alpha}_0$!].

It is then easily seen that (83) equations worked out in terms of the eight $(\varphi, E_a, \check{B}_a, \check{\varphi})$ are equivalent to

$$n = 4 \rightarrow \begin{cases} (\partial_t - i\kappa)\varphi - \partial_a E_a = 0 \\ (\partial_t + i\kappa) E_a - \partial_a \varphi - \varepsilon_{abc}\partial_b \check{B}_c = 0 \\ (\partial_t - i\kappa)\check{B}_a - \partial_a \check{\varphi} + \varepsilon_{abc}\partial_b E_c = 0 \\ (\partial_t + i\kappa)\check{\varphi} - \partial_a \check{B}_a = 0 \end{cases} \tag{95}$$

These equations duly imply that each component of $(\varphi, \ldots)$ is annulled by the operator $(\Delta - \kappa^2) = (-\partial_t^2 + \partial_a\partial_a - \kappa^2)$. This (complex) differential structure is very close to the (complex) Maxwell equations—which has motivated assumed notation. Indeed, (95) with $\kappa = 0$ and $\varphi = \text{const}$, $\check{\varphi} = \text{const}$, written in the elementary vectorial notation amount to

$$\begin{aligned} \partial_t \mathbf{E} - \operatorname{rot}\mathbf{B} = 0, \qquad & \partial_t \mathbf{B} + \operatorname{rot}\mathbf{E} = 0 \\ \operatorname{div}\mathbf{E} = 0, \qquad & \operatorname{div}\mathbf{B} = 0 \end{aligned} \tag{96}$$

Notice, however, that according to the identifications in (94), $(E_a, \check{B}_a)$ *do not* transform under the Poincaré group like components of $f_{\mu\nu} = f_{[\mu\nu]} \in \mathscr{C}$.

Of course, arranging $(\varphi, \ldots)$ in the form of a column with 8 complex entries, ψ, the equations (95) can be easily written in the form of (85), H being constructed from some 8×8 complex matrices, parallelly to the cases $n = 2, 3$. But in the considered case of $n = 4$ we can do better than that. Encode the 16 independent (real components of $\alpha_{\mu_1 \ldots \mu_p}$, $p = 0, \ldots, 4$) in the form of the complex 2×2 matrices,

$$Q := \begin{pmatrix} \varphi, & \bar{\check{\varphi}} \\ -\check{\varphi}, & \bar{\varphi} \end{pmatrix}, \qquad Q_a := \begin{pmatrix} E_a, & \bar{\check{B}}_a \\ -\check{B}_a, & \bar{E}_a \end{pmatrix} \tag{97}$$

Then, (95) equations—*together* with their complex conjugates—are easily seen equivalent to

$$\begin{aligned} [\partial_t - \kappa(i\sigma_3)] Q - \partial_a Q_a = 0 \\ [\partial_t + \kappa(i\sigma_3)] Q_a - \partial_a Q + (i\sigma_2)\varepsilon_{abc}\partial_b Q_c = 0 \end{aligned} \tag{98}$$

This opens doors for the *quaternionic* interpretation of the structure $(d + \delta + \kappa)\alpha = 0$—with the real flat $V_4 = (M, g)$ of signature $(+, -, -, -)$. Indeed, identify the basis of the quaternion algebra [see (86)] with 1 and

$$i := -i\sigma_1, \qquad j := i\sigma_2, \qquad k := i\sigma_3 \tag{99}$$

Then, as is well known, there are valid the crucial properties $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$ (and cyclically, with respect to i, j, k), while under the conjugation of quaternions, i, j, and k change sign, 1 remaining "real." This interpretation assumed, the (97) formulas can be understood as

$$\begin{aligned} Q &= \mathrm{Re}(\varphi).1 + I_m(\check{\varphi}).i + \mathrm{Re}(\check{\varphi}).j + I_m(\varphi).k \\ &= \alpha_0.1 - \check{\alpha}_t.j + \check{\alpha}_0.j + \alpha_t.k \\ Q_a &= \mathrm{Re}(E_a).1 + I_m(\check{B}_a).j + \mathrm{Re}(\check{\beta}_a).j + I_m(E_a)\cdot k \\ &= -\alpha_{at}.1 - \check{\alpha}_a.j - \tfrac{1}{2}\varepsilon_{abc}\alpha_{bc}.j + \alpha_a.k \end{aligned} \tag{100}$$

[Of course, representing the quaternions (Q, Q_a) in terms of the real components of α_0, α_μ, $\alpha_{\mu\nu}$, $\check{\alpha}_\mu$, and $\check{\alpha}_0$, (94) was used.] Define now 4×4 matrices with the quaternionic entries

$$A_1 := \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & j \\ 0 & 0 & j & 0 \end{array}\right), \qquad A_2 := \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -j \\ \hline 1 & 0 & 0 & 0 \\ 0 & j & 0 & 0 \end{array}\right)$$

$$A_3 := \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & j & 0 \\ \hline 0 & -j & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right), \qquad B := \left(\begin{array}{cc|cc} k & 0 & 0 & 0 \\ 0 & -k & 0 & 0 \\ \hline 0 & 0 & -k & 0 \\ 0 & 0 & 0 & -k \end{array}\right) \tag{101}$$

$$id := \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

These matrices are endowed with the properties that (i) the four (A_a, B) anticommute, (ii) $A_1^2 = A_2^2 = A_3^2 = id = -B^2$, and (iii) the A_a's are Hermitian with respect to "t" operation understood as simultaneous, transposition of matrices and the conjugation of quaternions, while B is anti-Hermitian.

Employing these concepts and the notions of

$$\psi := \begin{pmatrix} Q \\ Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}, \qquad \Psi := \left(\overline{Q}, \overline{Q}_1, \overline{Q}_2, \overline{Q}_3\right) \tag{102}$$

we can rewrite now our evolution equations (98) as

$$(\partial_t - \mathscr{H})\psi = 0 \Leftrightarrow \psi^+ \left(\overleftarrow{\partial}_t - \overleftarrow{\mathscr{H}}^+\right) = 0 \tag{103}$$

where

$$\mathscr{H} := A_a \partial_a + B\mathscr{H} \Rightarrow \overleftarrow{\mathscr{H}}^+ = A_a \overleftarrow{\partial}_a - B\mathscr{H} \tag{104}$$

From our equations stated in this form it follows at once

$$\partial_t(\psi^+\psi) + \partial_a \psi^+ A_a \psi = 0 \tag{105}$$

which with $\psi^+\psi = id.\rho$ provides an alternative proof of the conservation law for $\rho \geqslant 0$, instrumental for the probabilistic interpretation. Notice also that in the case of the considered four-dimensional flat (M, g) of signature $(+, -, -, -)$, ρ expressed in terms of components of the real bosonic fields

$$\rho = \alpha_0^2 + \check{\alpha}_0^2 + \alpha_t^2 + \alpha_a\alpha_a + \check{\alpha}_t^2 + \check{\alpha}_a\check{\alpha}_a + \alpha_{at}\alpha_{at} + (1/2)\alpha_{ab}\alpha_{ab} \tag{106}$$

amounts just to the sum of squares of all independent components of $\alpha = \oplus\alpha_p$.

Now, from a purely algebraic point of view (103) and (104) if postulated *a priori*, with anticommuting A_a's and B, A_a's Hermitian and B anti-Hermitian, can be interpreted as a specific realization of the Dirac idea of executing the root $\partial_t = (\partial_a \partial_a - \kappa^2)^{1/2} = A_a \partial_a + B\kappa$ in terms of 4×4 matrices. Realizing these matrices with complex entries one arrives at the standard Dirac equations for a fermionic particle of $s = 1/2$. Realizing the matrices with quaternionic entries as given in (101), one arrives at our scheme whose Poincaré group invariance is assured when the quaternionic components of ψ are constructed from the components of the real multiplet of the bosonic fields $(\alpha_0, \alpha_\mu, \check{\alpha}_{\mu\nu}, \check{\alpha}_0)$ according to the second lines of (100). The formal argument outlined above certainly supports our hypothesis that $\alpha = \oplus\alpha_p$ submitted to $(d + \delta + \kappa)\alpha = 0$ deserves the rank of a *natural* multiplet of the bosonic dynamical fields endowed with the same mass.

It should be observed, however, that from the point of view of results (and their interpretation) of Sections 3 and 4, the material of the present section is not very essential and can be viewed as a "bagatelle" only. Altogether, it amounts only to a study of the—already known to be relevant—invariant evolution equation $(d+\delta+\kappa)\alpha=0$ in terms of convenient linear combinations of components of $\alpha_{\mu_1\ldots\mu_p}$, which allow for the "*á la* Dirac" interpretation of the evolution process.[5]

6. CONCLUDING REMARKS

This paper is meant as a preliminary report only on the basic implications of the general hypotheses outlined in the Introduction. We have seen that $(d+\delta+\kappa)\alpha=0$ equations for $n=4$, signature $(+,-,-,-)$ constitute a rich, and natural, dynamical structure. The structure "per se," at least in case $V_4=V_1\times V_3$, allows a natural probabilistic interpretation. This interpretation although "unorthodox" from the point of view of the standard quantum mechanics, relies on the existence of $P_{\mu\nu}$ algebraically constructed from the dynamical fields which fulfills $P_{\mu\nu}{}^{;\nu}=0$, and, oddly enough, fulfills exactly the *minimal* requirements necessary for $\alpha=\oplus\alpha_p$ to be understood as a quantum mechanical set of states [see Mielnik (1969), particularly the comments on p. 41).] Although in the body of this paper the second quantization aspect of the structure $(d+\delta+\kappa)\alpha=0$ has not been studied, it is perhaps of interest to mention that the (flat) model case of $n=2$, signature $(+,-)$, chart $x^\mu=(t,x)$, with the Lagrangian from (39), when submitted to the routine second quantization for the bosonic fields—i.e., with commutators up to Dirac's δ's—does behave consistently. With the Hamiltonian operator $H=\int dx\,T_{tt}$, T_{tt} being component of $T_{\mu\nu}$ from (42), the $(d+\delta+\kappa)\alpha=0$ equations hold—understood as the Heisenberg equations, but diagonalized H has indefinite sign. The canonical second quantization of the "realistic" flat case with signature $(+,-,-,-)$ conditions (*á la* Proca) $\alpha_0|\ \rangle=0$, $\check{\alpha}_0|\ \rangle=0$ are likely to lead to the positive definite energy.

We confined this paper to a study of the free field $\alpha=\oplus\alpha_p$ only. It is clear, however, that the possible physical relevance of its ideas ought to be

[5]Some bolder interpretations of quaternionic equations (103) of the type, e.g., either (i) allowing for the transformations $\psi'=e^T\psi$, $T^+=-T\Rightarrow\rho'=\rho$, or (ii) attempting to interpret ψ's as "states" which could be superposed with quaternionic coefficients, are outside the scope of this paper. Generalizations of these types would not leave invariant (100) equations which constitute the bridge between the quaternionic treatment of the evolution process and our basic Poincaré invariant $(d+\delta+\kappa)\alpha=0$ equations. For a quantum mechanical interpretation of the Hilbert space over quaternions see Finkelstein et al. (1962).

studied with the bosonic multiplet α interpreted as the carrier of interactions among the fermions. The free evolution of particles states—which in our case mixes the different spin states of α—is not very essential, because in practice one always works in the interaction picture, where that evolution is precisely eliminated.

Leaving the basic problem open, we will contribute at this point two remarks only. First, according to the equivalence (59), if α_μ and $\check{\alpha}_\mu$ which fulfill K. G. equations are of the same order and κ is big, then the α_0, $\check{\alpha}_0$ and $\alpha_{\mu\nu}$ members of the D_4 structure are smaller by the factor P/κ, P being the order of magnitude of the linear momentum. Thus, in the limit $P/\kappa \ll 1$, only α_μ and $\check{\alpha}_\mu$ would effectively participate in a hypothetical interaction of $\alpha = \oplus\alpha_p$ with the fermions. This can be perhaps understood as a hint that our dynamical scheme can be of interest from the point of view of the phenomenological "V–A" theory of weak interactions. Second, we observe that given a spin $=1/2$ spinor ψ, the natural "sources" for our bosonic multiplet α, $(d+\delta+\kappa)\alpha = j$, can be guessed to have the shape

$$\begin{aligned} j = \psi^+ \{ & g_s + g_v \gamma_\mu dx^\mu + g_t (1/2)\gamma_{\mu\nu} dx^\mu \wedge dx^\nu \\ & + \check{g}_v * \gamma_5 \gamma_\mu dx^\mu + \check{g}_s * \gamma_5 \} \psi \end{aligned} \tag{107}$$

where g's are coupling constants.

APPENDIX: A SPHERICALLY SYMMETRIC SOLUTION

An explicit spherically symmetric solution to $(d+\delta+\kappa)\alpha = 0$ for the (flat) case of $n=4$, signature $(t,-,-,-)$ can be easily constructed. Indeed, specialize our equations in the form (95) for the case of

$$\begin{aligned} E_a &= \partial_a r^{-1} e(t,r), & \check{B}_a &= \partial_a r^{-1} \check{g}(t,r) \\ \varphi &= r^{-1} h(t,r), & \check{\varphi} &= r^{-1} \check{h}(t,r) \\ & & r &:= (x^a x^a)^{1/2} \end{aligned} \tag{A1}$$

Then, because $\partial_a \partial_a$ acting on (t,r) dependent objects amounts to $r^{-1}\partial_r^2 r$, and taking into account that the definitions of e and $\check{g}$ allows the gauge $e \to e + k(t)\cdot r$, $\check{g} \to \check{g} + \check{k}(t)\cdot r$, with $k, \check{k}$ which can be chosen as convenient, one easily infers that (95) equations became equivalent to

$$\left.\begin{aligned} (\partial_t - i\kappa)h - \partial_r^2 e = 0 \\ (\partial_t + ti\kappa)e - h = 0 \end{aligned}\right\} \Rightarrow (-\partial_t^2 + \partial_r^2 - \kappa^2)e = 0$$

$$\left.\begin{aligned} (\partial_t - i\kappa)\check{g} - \check{h} = 0 \\ (\partial_t + i\kappa)\check{h} - \partial_r^2 \check{g} = 0 \end{aligned}\right\} \Rightarrow (-\partial_t^2 + \partial_r^2 - \kappa^2)\check{g} = 0 \tag{A2}$$

Therefore, the general solutions of the (formal) for $n = 2$ K. G. equations, $e(t, r)$ and $\check{g}(t, r)$, determine entirely the spherically symmetric solution to $(d + \delta + \kappa)\alpha = 0$.

Constraining further this solution to be static, i.e., somewhat analogous to the Coulomb field in the case of Maxwell equations, we arrive at

$$\begin{aligned} e &= e_0 e^{-\kappa r}, & \check{g} &= \check{g}_0 e^{-\kappa r} \\ h &= i\kappa e_0 e^{-\kappa r}, & \check{h} &= i\kappa \check{g}_0 e^{-\kappa r} \end{aligned} \tag{A3}$$

where e_0 and $\check{g}_0$ are (complex) constants. The sign of exponents have been chosen as minus, anticipating that it will help for convergence of $\langle \alpha, \alpha \rangle = \int \sin\theta \, d\theta \, d\varphi \, r^2 \, dr \rho$, with

$$\rho = \bar{\varphi}\varphi + \bar{E}_\mu E_a + \bar{\check{B}}_a \check{B}_a + \bar{\check{\varphi}}\check{\varphi} \tag{A4}$$

However, with (A3), $\langle \alpha, \alpha \rangle = -4\pi(|e_0|^2 + |\check{g}_0|^2) \cdot \int_0^\infty d\{r^{-1}(1 + Fr)e^{-2xr}\}$ turns out to be divergent like $-\int_0^\infty d(1/r) = +\infty$. This is somewhat analogous to the divergence of $\int d_3 \times \bar{\psi}\psi$ in the case of the static spherically symmetric solution to $[-(\hbar^2/2m)\partial_a\partial_a - i\hbar\partial_+]\psi = 0$.

It would be perhaps of some interest to examine on the level of general relativity $[n = 4$, signature $(+, -, -, -)]$, the simultaneous solution to $G_{\mu\nu} = \frac{8\pi G}{C^4}\Upsilon_{\mu\nu} - \Upsilon_{\mu\nu}$ defined by (42)—and $(d + \delta + \kappa)\alpha = 0$ in the case of the spherical symmetry: the resulting partial differential equations in (t, r) appear to be reasonably manageable.

NOTE ADDED IN PROOF

The equation $(d + \delta + x)\alpha = 0$ was studied first by E. Kähler (1962), Rendiconti di Mat. (Roma) Ser V, 21. In its aspect of the Dirac–Kähler algebra, the equation as interpreted in terms of fermionic fields was discussed recently by P. Becher, Phys. Letters (1981), 104B, 221 and P. Becher and H. Joos: DESY 82-031 (1982).

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